MATH2050C Assignment 13

Section 5.6 no. 3, 4, 14, 15.

No need to hand in any problem.

Monotone Functions, Continuity and Their Inverse

We study monotone functions and their inverse. We will pay attention only to increasing ones, while the decreasing ones can be treated in a similar way. (Or observe that $-f$ is increasing when f is decreasing.)

First, we show that only jump discontinuity is admitted for monotone functions.

Proposition 1. Let f be an increasing function on some interval $[a, b]$. Then $\lim_{x\to x_0^+} f(x)$ and $\lim_{x\to x_0^-} f(x)$ always exist for every $x_0 \in (a, b)$.

Proof. Claim $\alpha \equiv \sup\{f(x) : x \in [a, x_0)\}$ is the left hand limit and $\inf\{h(x) : x \in (x_0, b]\}$ is the right hand limit. Since f is increasing, we have $f(x) \le f(b)$ which means α is a finite number. To prove it is the left hand limit of f at x_0, we need to show, for $\varepsilon > 0$, there is some δ such that $|f(x) - \alpha| < \varepsilon$ for $x \in (x_0 - \delta, x_0)$. By the definition of α, for $\varepsilon > 0$, there is some $f(x_1), x_1 < x_0$, such that $f(x_1) + \varepsilon > \alpha$. By monotonicity, it follows that $f(x) + \varepsilon \ge f(x_1) + \varepsilon > \alpha$ for all $x, x \in [x_1, x_0)$, so $|f(x) - \alpha| < \varepsilon$, done. The right hand limit can be treated in a similar manner.

As a consequence of this proposition, we have

Proposition 2. An increasing function f is continuous on $[a, b]$ if and only if the range of f is $[f(a), f(b)]$.

Proof. When f is continuous on $[a, b]$, its image is an interval (Preservation of Interval Theorem). Moreover, its maximum and minimum are attained (Max-Min Theorem). It follows that $f([a, b])$ is equal to $[m, M]$ where m and M are respectively the minimum and maximum of f. As f is increasing, $[m, M]$ is equal to $[f(a), f(b)]$. On the other hand, if f is not continuous at some $x_0 \in (a, b)$, $\alpha \equiv \lim_{x\to x_0^-} f(x) < \beta \equiv \lim_{x\to x_0^+} f(x)$ according to Proposition 1. Now, the set $(\alpha, \beta) \setminus \{f(x_0)\}$ lie outside the range of f, hence $[f(a), f(b)]$ cannot be an interval. The case of possible discontinuity at a or b can be treated similarly.

Corollary 3. An increasing function f is continuous on $(a, b), -\infty \le a < b \le \infty$, if and only if the range of f is (α, β) where $\alpha = \inf f((a, b))$ and $\beta = \sup f((a, b))$.

Proof. We pick $a_n \in \mathbb{R}$ decreasing to a and b_n increasing to b and then apply Proposition 2 to f on each $[a_n, b_n]$.

In the following we set $j_f(x_0) = f(x_0^+) - f(x_0^-)$ where f is increasing and $f(x_0^+) = \lim_{x\to x_0^+} f(x)$ and $f(x_0^-) = \lim_{x\to x_0^-} f(x)$. Note that f is continuous at x_0 if and only if $j_f(x_0) = 0$. (In general, for any function f, one may define $j_f(x_0) = |f(x_0^+) - f(x_0^-)|$ provided the one-sided limits limit. Then x_0 is a continuity point if and only if $f(x_0^+) = f(x_0)$ and $j_f(x_0) = 0$.)

Proposition 4. Let f be an increasing function on $[a, b]$. For any given number $\alpha > 0$, the set $\{x \in [a, b] : j_f(x) \ge \alpha\}$ is a finite set.

Proof. Pick N many points in this set, $x_N < x_{N-1} < \cdots x_2 < x_1$. By monotonicity,

$$\begin{aligned} f(b)-f(a) &= (f(b)-f(x_1^+))+(f(x_1^+)-f(x_1^-))+(f(x_1^-)-f(x_2^+))+ \\ &\quad (f(x_2^+)-f(x_2^-))+((f(x_2^-)-f(x_3^+))+\cdots+(f(x_N^-)-f(a)) \\ &\geq (f(x_1^+)-f(x_1^-))+(f(x_2^+)-f(x_2^-))+\cdots+(f(x_N^+)-f(x_N^-)) \\ &\geq N\alpha \ . \end{aligned}$$

It follows that $N \leq (f(b)-f(a))/\alpha$, that is, there are no more than $(f(b)-f(a))/\alpha$ many points in this set. Hence this set is finite for each given α.

Theorem 5. An monotone function on $(a,b), -\infty \leq a < b \leq \infty$, has at most countably many points of discontinuity.

Proof. Assume f is increasing on $[a,b], a,b \in \mathbb{R}$ first. Let $E_j = \{z \in [a,b] : \lim_{x\to z^+} f(x) - \lim_{x\to z^-} f(x) \geq 1/j\}$. By Proposition 1, any discontinuity of f belongs to some E_j. Therefore, the discontinuity set which is equal to $\cup_{j=1}^{\infty} E_j$ is a countable set. (The countable union of countable sets is a countable set.)

When f is increasing on $(a,b), -\infty \leq a < b \leq \infty$, we pick a_n decreasing to a and b_n increasing to b and apply the previous paragraph to f on $[a_n, b_n]$ to conclude that the set $D_n = \{x \in [a_n, b_n] : \ f \text{ is discontinuous at } x\}$ is countable for each n. Therefore, the discontinuity set of f over (a,b), which is the countable union of all D_n over n, is again a countable set.

In fact, Theorem 5 is valid for all monotone functions on any interval. You may modify the proof here or there to suit all different cases.

Now we establish the continuity of the inverse of a continuous, strictly increasing function.

Theorem 6. Let f be a continuous, strictly increasing (resp. strictly decreasing) function on (a,b). Its inverse function f^{-1} is a continuous, strictly increasing (resp. strictly decreasing) function on (α,β) where $\alpha = \inf\{f(x) : \ x \in (a,b)\}$ and $\beta = \sup\{f(x) : \ x \in (a,b)\}$.

Proof. The inverse function clearly exists and is strictly increasing. It suffices to show that it is continuous. Let $y_0 \in (\alpha,\beta)$ and $x_0 = f^{-1}(y_0)$. We claim that $\lim_{y\to y_0^+} f^{-1}(y) = x_0$. Let $\{y_n\} \to y_0$ from the right, we need to show $\lim_{n\to\infty} f^{-1}(y_n) = x_0$. To do this, fix two points $y_1 < y_0 < y_2$ in the interval so that $y_1 \leq y_n \leq y_2$ for all n. Then $f^{-1}(y_1) \leq f^{-1}(y_n) \leq f^{-1}(y_2)$ shows that $\{f^{-1}(y_n)\}$ is a bounded sequence, so by Bolzano-Weierstrass Theorem it has a convergent subsequence $f^{-1}(y_{n_j}) \to z_0$. By the continuity of f, $f(f^{-1}(y_{n_j})) \to f(z_0)$ which means $f(z_0) = y_0$. It follows that $z_0 = x_0$. Now, for $\varepsilon > 0$, there is some j_0 such that $|f^{-1}(y_{n_j}) - x_0| < \varepsilon$ for all $n_j \geq n_{j_0}$. (In fact, it is $0 \leq f^{-1}(y_{n_j}) - x_0 < \varepsilon$.) As $y_n \to y_0$ from the right hand side, we can find a large n_1 such that $y_0 \leq y_n \leq y_{n_{j_0}}$ for all $n \geq n_1$. Then $0 \leq f^{-1}(y_n) - x_0 \leq f^{-1}(y_{n_{j_0}}) - x_0 < \varepsilon$ for all $n \geq n_1$, that is, $\lim_{n\to\infty} f^{-1}(y_n) = x_0$.

Similarly, we can show that $\lim_{y\to y_0^-} f^{-1}(y) = x_0$. Hence, f^{-1} is continuous at y_0.

As an application, consider the function $f(x) = x^n$ where $n \in \mathbb{N}$. It is routine to check that it is strictly increasing on $[0,\infty)$. By Theorem 6, its inverse function f^{-1} is continuous from $[0,\infty)$. In fact, when n is odd, f is strictly increasing on $(-\infty,\infty)$ so the inverse function exists on $(-\infty,\infty)$. We use the notation $x^{1/n}$ to denote $f^{-1}(x)$, so $(x^{1/n})^n = x$ and $(x^n)^{1/n} = x$ which means $f(f^{-1}(x) = x$ and $f^{-1}(f(x)) = x$ respectively holds on $[0,\infty)$.

When n is a negative integer, x^n is continuous, strictly decreasing on $(0, \infty)$ and its inverse $x^{1/n}$ is a continuous strictly decreasing function on $(0, \infty)$.

When $r = m/n$ is a rational number, we define its the r**-th power** by $x^{m/n} = (x^{1/n})^m$. It is a continuous function on $(0, \infty)$. We refer to the text book for properties of this function.